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A new kind of solitary wave

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The investigation focuses on solitary-wave solutions of an approximate pseudo-differential equation governing the unidirectional propagation of long waves in a two-fluid system where the lower fluid with greater density is infinitely deep and the interface is subject to capillarity. The validity of this model equation is shown to depend on the assumption that $T/g(\rho_2-\rho_1)h^2 \gg 1$, where T is the interfacial surface tension, $\rho_2-\rho_1$ the difference between the densities of the fluids and h the undisturbed thickness of the upper layer.

Various properties of solitary waves are demonstrated. For example, they have oscillatory outskirts and their velocities of translation are less than the minimum velocity of infinitesimal waves. Also, they realise respective minima of an invariant functional for fixed values of another such functional, being in consequence orbitally stable. Explicit non-trivial solutions of the equation in question are unavailable, but an existence theory is presented covering both periodic and solitary waves of permanent form.

1. Introduction

Attention has recently been given to the possibility of gravity–capillary surface waves of solitary type on deep water. Such solitary waves have speeds of translation less than the minimum speed of infinitesimal waves, namely $(4gT/\rho)^{\frac{1}{4}}$, where T is the surface tension and ρ the density of water. (Thus for water the minimum speed is about 23 cm s^{-1}.) Extensive investigations into the properties of these waves have been reported by Longuet-Higgins (1989) and Vanden-Broeck & Dias (1992). But it so far remains an open question whether or not such solitary waves are stable.†

The present paper deals with a comparable class of solitary waves about which definite theoretical evidence of stability is available. The waves can occur in a two-fluid system where a thin layer of incompressible fluid with density ρ_1, bounded above by a rigid horizontal plane, lies on a very deep incompressible fluid with density $\rho_2 > \rho_1$. It will be shown in §2 that, provided the interfacial surface tension T satisfies $T \gg g(\rho_2-\rho_1)h^2$, where h is the undisturbed thickness of the upper layer, long waves of small amplitude propagating unidirectionally in this system are governed approximately by the (dimensionless) equation

$$u_t + u_x + 2uu_x - \alpha \mathrm{L} u_x - \beta u_{xxx} = 0, \tag{1.1}$$

in which α and β are positive constants, with $\beta \gg \alpha$ for consistency of the approximation. Here L is the linear symmetric pseudo-differential operator defined by its symbol $|k|$ (in the sense that $-\partial_x^2$ has the symbol k^2). The same operator appears in the Benjamin–Ono equation (named after Benjamin 1967 and Ono 1975).

Among properties of solitary-wave solutions of (1.1) to be examined in §3, a variational principle for them will subsequently be the most helpful. Specifically, any

† And, as far as I know, there is yet no experimental observation of them.

such solution in the form $u = \phi(x-ct)$ realizes the (negative) minimum of a functional G for a respective given value of another, positive-definite functional F, both of which functionals are invariant for any solution u of (1.1). This attribute strongly suggests that solitary-wave solutions are *orbitally stable* (i.e. stable in respect of shape and size), although a proof of stability will not be included here.

The variational principle will be used in §4 to prove that (1.1) has smooth solitary-wave solutions $\phi(x-ct)$, and also steady-wave solutions $\phi(x-ct; 2l)$ that are periodic with sufficiently large period $2l$. Either class of solutions can be parameterized by values of $F > 0$, and it will be shown that $\mathrm{d}c/\mathrm{d}F < 0$.

When $\alpha = 0$, equation (1.1) reduces to the well-known equation of Korteweg & de Vries (1895). But, as was noted in their original paper and as was re-emphasized by Benjamin (1982), when $\beta > 0$ solitary-wave solutions of the KdV equation have properties qualitatively different from those when $\beta < 0$. (As regards the original application to long water waves in an open channel of depth H, the present case $\beta > 0$ corresponds to Bond number $\tau = T/\rho g H^2 > \frac{1}{3}$.) With $\alpha = 0$ and $\beta > 0$ in (1.1), these solutions are

$$\phi = -\frac{3}{2}\left(\frac{1-c}{\beta}\right)\operatorname{sech}^2\left\{\frac{1}{2}\left(\frac{1-c}{\beta}\right)^{\frac{1}{2}}(x-ct)\right\}, \tag{1.2}$$

where necessarily $c < 1$. Thus $\phi(x) < 0$ for all $x \in \boldsymbol{R}$, and c is less than the minimum velocity of infinitesimal waves (i.e. dimensionally $c < (gH)^{\frac{1}{2}}$ in the original application with $\tau > \frac{1}{3}$).

The well-known result (1.2) represents an extreme case of the present class of solitary waves, being of course recovered in the limit $\alpha \downarrow 0$. But explicit steady-wave solutions of (1.1) are unknown in the case of both α and β positive. Solitary-wave and periodic solutions of (1.1) are known explicitly in the case $\beta = 0$ (Benjamin 1967), solitary waves then being positive with $c > 1$ if $\alpha > 0$. It would be misleading, however, to regard this case too as being covered by the present theory, which depends crucially on the specification that $\beta > 0$.

It should be acknowledged that oscillatory solitary waves akin to the present ones have also been studied theoretically by Iooss & Kirchgässner (1990), who reappraised the classic open-channel model in the case $\tau < \frac{1}{3}$. Their problem is in effect the finite-depth counterpart of the problem treated by Longuet-Higgins (1989) and Vanden-Broeck & Dias (1992). As in the present case, the speed of infinitesimal waves then has a minimum less than the speed of infinitesimal waves of extreme length, and there appear to be solitary waves with speeds less than this minimum. Iooss & Kirchgässner examined the respective mathematical problem from the standpoint of bifurcation theory; but they too offered no conclusion about the stability and consequent realizability of the solitary waves in question. The present problem is quite different in detail from theirs and the other cited.

2. Derivation of evolution equation

Long-wave models such as the KdV equation, representing the interaction of small nonlinear and dispersive effects on unidirectional waves, have been derived by a variety of more or less equivalent means (e.g. Benjamin 1967, §§1 and 2; Whitham 1974, §13.11). Here it is apposite merely to highlight the essential ingredients, rather than completing the details of a formal derivation.

Let us start from the dispersion relation that holds for infinitesimal waves in the two-fluid system proposed in §1. Incompressible inviscid fluid of density ρ_1 lies in a layer of thickness h above an infinitely deep incompressible inviscid fluid of density $\rho_2 > \rho_1$. The less-dense fluid is bounded above by a rigid horizontal plane and the interface between the fluids is subject to surface tension T. One supposes that the vertical displacement of the interface has the form $\eta = \epsilon \exp\{\mathrm{i}(\omega t - kx + b)\}$, where ω, k and b are real constants, ϵ is an infinitesimal constant, x is a horizontal coordinate and t is time. Then, by standard methods (cf. Lamb 1932, §231), the phase velocity $c = \omega/k$ is found to be related to wavenumber k according to

$$c^2 = \frac{g(\rho_2-\rho_1)+Tk^2}{\rho_1 k \coth kh + \rho_2|k|}. \tag{2.1}$$

Provided the effect of surface tension is included additionally, this formula is recovered by taking the limit $h' \to \infty$ in a result given by Lamb (1932, p. 371, eqn. (11)), which allows for the lower fluid to be bounded by a rigid horizontal plane at depth h' below the interface.

In the limit $|k| \to 0$, (2.1) shows that the speed c_0 of extremely long infinitesimal waves is the (positive) square root of

$$c_0^2 = gh\left(\frac{\rho_2}{\rho_1} - 1\right).$$

The value $c(0) = c_0$ is a local maximum of $c(k)$, but is plainly not an absolute maximum if $T > 0$.

The relation $c = c(k)$ is not analytic at the origin, which fact has been noted in a recent paper (Benjamin 1993*a*) to exemplify a common peculiarity of hydrodynamic models that feature an infinite expanse of incompressible fluid. To provide the basis of a long-wave theory, however, the even function $c(k)$ can be approximated by the leading terms of its expansion in powers of $|k|$. Thus, from (2.1), for waves propagating in the positive x-direction ($c > 0$), we deduce that

$$c = c_0\left[1 - \frac{1}{2}\frac{\rho_2}{\rho_1}h|k| + \frac{1}{2}\left\{\frac{T}{g(\rho_2-\rho_1)} + \left(\frac{3}{4}\frac{\rho_2^2}{\rho_1^2} - \frac{1}{6}\right)h^2\right\}k^2 + O(|k|^3)\right] \tag{2.2}$$

as an asymptotic approximation for small $|k|$. For consistency in what follows, it is required that $k_{\mathrm{m}} h \ll 1$, where k_{m} is the value of $|k|$ minimizing c as given by (2.2) (i.e. the minimizing wavelength is much greater than the thickness h of the upper layer). Thus, on the assumption that ρ_2 is not much different from ρ_1, the relevant case is where

$$2T/g(\rho_2-\rho_1)h^2 \gg 1. \tag{2.3}$$

Hence, when h is adopted as the unit of length and h/c_0 as the unit of time, the approximate dispersion relation takes the dimensionless form

$$c = 1 - \alpha|k| + \beta k^2, \tag{2.4}$$

in which $\alpha = \tfrac{1}{2}\rho_2/\rho_1, \quad \beta = \tfrac{1}{2}T/g(\rho_2-\rho_1)h^2.$

Note that c has a minimum $c_{\mathrm{m}} = 1 - \tfrac{1}{4}\alpha^2/\beta$ at $|k| = k_{\mathrm{m}} = \tfrac{1}{2}\alpha/\beta$, and that according to (2.3) both k_{m} and $1 - c_{\mathrm{m}} > 0$ are small in comparison with 1.

It is thus implied that all sinusoidal long waves of infinitesimal amplitude propagating in the x-direction satisfy approximately the dimensionless linear equation

$$\eta_t + (\eta - \alpha \mathrm{L}\eta - \beta\eta_{xx})_x = 0, \tag{2.5}$$

because $|k|$ is the symbol of L and k^2 the symbol of $-\partial_x^2$. Therefore, by the Fourier principle, equation (2.5) also governs the unidirectional propagation of infinitesimal long waves of arbitrary form; and the same equation is satisfied by velocity components and other variables describing the motion, all of which are linearly related to the interfacial displacement $\eta(x,t)$.

The terms with coefficients α and β in (2.5) account approximately for dispersive effects on reasonably long waves, being then small corrections to

$$\eta_t + \eta_x = 0, \tag{2.6}$$

which is one factor of the wave equation $\eta_{tt} = \eta_{xx}$. Specifically, if the (dimensional) lengthscale of the wave motion is $\lambda \gg h$, then the dispersive terms in (2.5) are $O(h/\lambda)$ relative to the leading terms copied in (2.6).

For long waves of finite but reasonably small amplitude, nonlinear effects are similarly accountable by a small correction to (2.6). Ignoring all dispersive effects, one obtains a nonlinear hyperbolic system equivalent to the shallow-water equations (Lamb 1932, §187). Hence, as is standard, consideration of simple waves propagating in the x-direction towards a state of rest will lead to the one-dimensional hyperbolic equation

$$u_t + u_x + 2uu_x = 0 \tag{2.7}$$

for a suitably normalized dependent variable $u(x,t)$.

Suppose the typical magnitude of u to be $a/h \ll 1$ and to be comparable with h/λ (i.e. $a\lambda/h^2 = O(1)$ as $h/\lambda \to 0$). Then one can infer that a consistent first approximation for both dispersive and nonlinear effects is given by adding the extra terms in (2.5) and (2.7) compared with (2.6). Thus the equation governing long waves subject to comparably small effects of dispersion and nonlinearity is

$$u_t + (u + u^2 - \alpha \mathrm{L}u - \beta u_{xx})_x = 0, \tag{2.8}$$

which is the same as (1.1).

More specifically, the scheme of approximation just outlined shows that the dependent variable in (2.8) is $u = \frac{3}{4}\bar{u}/c_0$, where $\bar{u}$ is the (dimensional) mean horizontal velocity in the upper layer. To the same order of approximation, it also appears that $u = \frac{3}{4}\eta/h$, where η is the downward vertical displacement of the interface.

Note that the surface-wave problem cited in the first paragraph of §1 is not recoverable by taking the limit $\rho_1 \to 0$ in the foregoing approximate analysis. The gravity–capillary surface solitary waves mentioned in §1 have no direct relation to the interfacial solitary waves described by (2.8). Although the limit $\rho_1 \to 0$ in (2.1) recovers the dispersion relation for infinitesimal surface waves, there is then no definable long-wave speed c_0 and long-wave approximations are irrelevant. In fact, nonlinear effects enter the surface-wave problem in ways quite different from those represented in (2.8).

3. Solitary-wave properties

It will be confirmed in §4 that (2.8) has solutions in the form $u = \phi(x-ct)$, where $\phi(x)$ is an even function satisfying

$$\partial_x^n \phi(x) \to 0 \quad \text{as} \quad x \to \pm\infty, \quad n = 0, 1, 2, \ldots. \tag{3.1}$$

In view of the property (3.1), an integration of (2.8) with respect to x after ϕ has been substituted shows the equation for ϕ to be

$$C\phi = \phi^2 - \alpha \mathrm{L}\phi - \beta\phi_{xx} \tag{3.2}$$

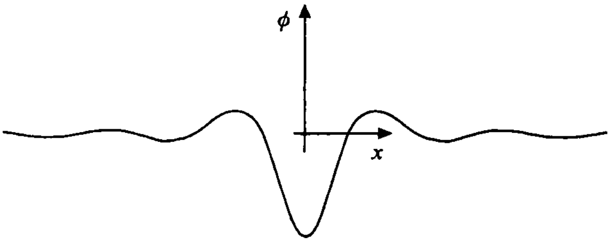

FIGURE 1. Sketch of waveform.

with $C = c-1$. Each term of this equation is an even function when ϕ is so. Because L and ∂_x^2 are translation-invariant operators, every translation $\phi(x-ct)$ satisfies (3.2) when $\phi(x)$ is a solution. Accordingly, ϕ can be treated as an even function and its significance as a steady travelling wave left implicit henceforth.

To provide the asymptotic property (3.1), the linearization of (3.2) must have solutions converging to zero as $x \to \pm\infty$. Postulating solutions $\exp(\mathrm{i}kx)$ with k complex, we find the possible values of k from (2.4) considered as a quadratic equation in k. It thus appears that

$$k = k_{\mathrm{m}} \pm \mathrm{i}\left(\frac{C_{\mathrm{m}}-C}{\beta}\right)^{\frac{1}{2}}, \tag{3.3}$$

where $k_{\mathrm{m}} = \frac{1}{2}\alpha/\beta$ and $C_{\mathrm{m}} = c_{\mathrm{m}}-1 = -\frac{1}{4}\alpha^2/\beta$ as noted below (2.4). This result shows that solitary waves must have $C < C_{\mathrm{m}}$: that is, their speeds of propagation are less than the minimum speed of infinitesimal sinusoidal waves. Solitary waves have oscillatory outskirts, where the spacing between successive zeros is $\pi/k_{\mathrm{m}} = 2\pi\beta/\alpha$; the ambiguous sign in (3.3) allows for exponential decay at a rate $[(C_{\mathrm{m}}-C)/\beta]^{\frac{1}{2}} > 0$ both as $x \to \infty$ and as $x \to -\infty$. Figure 1 shows a sketch of the waveform.

From (3.2) and the asymptotic property (3.1) of ϕ, it follows that

$$C\int_R \phi\,\mathrm{d}x = \int_R \phi^2\,\mathrm{d}x,$$

which, as $C < C_{\mathrm{m}} < 0$, shows that

$$\int_R \phi\,\mathrm{d}x < 0. \tag{3.4}$$

It also follows that

$$C\int_R \phi^2\,\mathrm{d}x = \int_R (\phi^3 - \alpha\phi\mathrm{L}\phi + \beta\phi_x^2)\,\mathrm{d}x. \tag{3.5}$$

Because $-\alpha|k| + \beta k^2 \geqslant -\frac{1}{4}\alpha^2/\beta = C_{\mathrm{m}}$ for real k, Parseval's theorem establishes that

$$\int_R (-\alpha\phi\mathrm{L}\phi + \beta\phi_x^2)\,\mathrm{d}x \geqslant C_{\mathrm{m}}\int_R \phi^2\,\mathrm{d}x.$$

Hence

$$\int_R \phi^3\,\mathrm{d}x \leqslant (C-C_{\mathrm{m}})\int_R \phi^2\,\mathrm{d}x < 0, \tag{3.6}$$

and consequently

$$\inf_{x\in R}\phi(x) \leqslant C - C_{\mathrm{m}} < 0. \tag{3.7}$$

In keeping with the properties (3.4), (3.6) and (3.7), it may be expected that even solitary-wave solutions of (3.2) satisfy $\phi(0) = \inf \phi(x) < 0$, as shown in figure 1. Moreover, the solitary-wave amplitude $-\phi(0)$ may be expected to increase with $C_{\mathrm{m}} - C$.

3.1. *Invariants*

In order to introduce further information about solitary waves, we now recognize properties attributable to all solutions $u(x,t)$ of the evolution equation (2.8). First note that

$$F(u) := \int \tfrac{1}{2} u^2 \, \mathrm{d}x = \text{const.} \tag{3.8}$$

Here and in what follows the range of integration is left implicit: either it is $\boldsymbol{R}$ in the case of solutions u that vanish together with all their x-derivatives as $x \to \pm\infty$, or it is a fixed interval $[-l, l]$ in the case of solutions that are periodic in x with period $2l$. As an operation on periodic functions, L has an obvious generalization in terms of Fourier series (see context of (4.4) in §4).

The property $\mathrm{d}F(u)/\mathrm{d}t = 0$ follows immediately upon differentiation of the integral (3.8) with respect to t and substitution for u_t from (2.8). In either of the cases just specified, the resulting integral is seen to vanish according to integrations by parts and the facts that L is symmetric and commutes with ∂_x.

Next note that

$$G(u) := \int (\tfrac{1}{3} u^3 - \tfrac{1}{2} \alpha u \mathrm{L} u + \tfrac{1}{2} \beta u_x^2) \, \mathrm{d}x = \text{const.} \tag{3.9}$$

Denoted by ∇, say, the variational derivative of $F(u) + G(u)$ recovers an expression appearing in (2.8), thus

$$\nabla (F+G)(u) = u + u^2 - \alpha \mathrm{L} u - \beta u_{xx}.$$

Hence, in view of (3.8), the property $\mathrm{d}G(u)/\mathrm{d}t = 0$ follows at once. In fact, $F+G$ is the (conserved) Hamiltonian functional for a Hamiltonian representation of (2.8), namely

$$u_t = -\partial_x \nabla (F+G)(u),$$

with cosymplectic operator $-\partial_x$ (cf. Benjamin **1993***b*, chaps. 3 and 4).

3.2 *Variational principle*

Equation (3.2) can be rewritten in the form

$$C \nabla F(\phi) = \nabla G(\phi), \tag{3.10}$$

being satisfied by solitary waves and by a kindred class of periodic steady waves to be considered in §4. A variational characterization of all such waves is thus indicated. The principle is

$$G(\phi) = \min G(u) \quad \text{for given} \quad F(u) > 0. \tag{3.11}$$

Plainly (3.10) is the Euler–Lagrange equation which is a necessary condition for ϕ to be the conditional minimizer, and C is the Lagrange multiplier. The spectrum of possible solitary waves corresponds to the prescribable values of $F(\phi)$.

Relative to any extremal ϕ, the conditional second variation of G for prescribed F is

$$\begin{aligned} S(\phi;\xi) &= \tfrac{1}{2}\epsilon^2 \left[\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2} \{ G(\phi+\epsilon\xi) - CF(\phi+\epsilon\xi) \} \right]_{\epsilon=0} \\ &= \epsilon^2 \int \{ (\phi - \tfrac{1}{2}C)\,\xi^2 - \tfrac{1}{2}\alpha \xi \mathrm{L} \xi + \tfrac{1}{2}\beta \xi_x^2 \} \, \mathrm{d}x, \end{aligned} \tag{3.12}$$

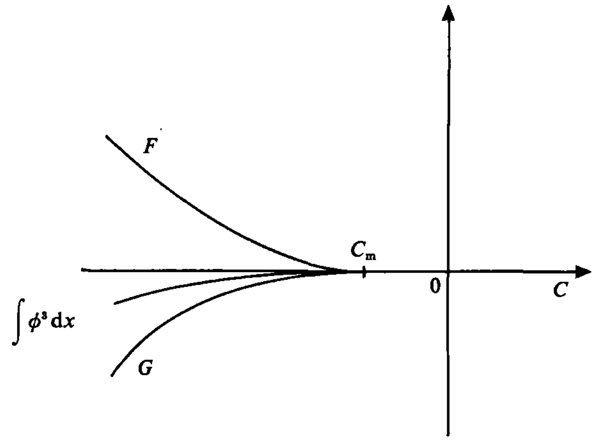

FIGURE 2. Dependence of $F(\phi)$ and $G(\phi)$ on $C < C_m < 0$.

in which the otherwise arbitrary variation $\xi(x)$ is subject to the (isoperimetric) condition

$$\int \phi \xi \, dx = 0. \tag{3.13}$$

For $G(\phi)$ to be a conditional minimum, the quadratic form $S(\phi;\xi)$ in ξ must be non-negative when subject to (3.13).

Note the case $\xi = \phi_x$, which satisfies (3.13). By differentiation of (3.2), it appears that $S(\phi;\phi_x) = 0$. But this case merely represents an infinitesimal translation of ϕ, which obviously cannot change G or F. Note also that, according to (3.2) and (3.12),

$$S(\phi;\phi) = \epsilon^2 \int \tfrac{1}{2}\phi^3 \, dx < 0.$$

Thus the condition (3.13) excluding the case $\xi = \phi$ is essential: $G(\phi) - CF(\phi)$ is a stationary value but cannot be an unconditional minimum.

3.3. *Dependence on C*

Consider the class of solitary waves ϕ parameterized by $C = c-1 < C_m < 0$. Because ϕ satisfies (3.2) (equivalently (3.10)), it is evident that

$$C \, dF(\phi)/dC = dG(\phi)/dC; \tag{3.14}$$

and it can be presumed that $G(\phi) \uparrow 0$ and $C - C_m \uparrow 0$ as $F(\phi) \downarrow 0$. (This assumption will be borne out by estimates to be introduced in §4.) Accordingly, (3.14) indicates

$$dF(\phi)/dC < 0, \quad dG(\phi)/dC > 0, \tag{3.15a, b}$$

as may be expected. Moreover, as (3.5) corresponds to

$$CF(\phi) - G(\phi) = \frac{1}{6}\int_R \phi^3 \, dx,$$

(3.14) implies

$$\frac{d}{dC}\left(\frac{1}{6}\int_R \phi^3 \, dx\right) = F(\phi) > 0, \tag{3.16}$$

which accords with (3.6) and (3.15*b*). The situation thus exposed is illustrated in figure 2.

With the integrals appropriately redefined, the properties (3.15) and (3.16) hold also for the class of periodic solutions $\phi(x;2l)$ with period $2l$ that will be established in §4. Perhaps needless to say, each class of steady-wave solutions can be considered to arise by bifurcation from the null solution of (3.2) as F is increased from zero, or as C is lowered through the value C_m.

4. Existence theory

The variational principle (3.11) will now be used to show that (3.2) has solitary-wave solutions. Having been applied previously to a closely related problem (Benjamin 1974, §4), the argument will be presented in essential details only.

4.1. *Periodic solutions*

Equation (3.2) is first shown to be satisfied *weakly* by even non-trivial functions $\phi(x;2l)$ that are periodic with period $2l$. Each of these functions realizes the minimum of $G(u)$ for a respective $F(u) > 0$ (with the integrals from $-l$ to l in the definitions (3.8) and (3.9) of F and G). Let us take $l = N\pi/k_{\mathrm{m}} = 2N\pi\beta/\alpha$ with $N = 1, 2, 3, \ldots$, so that, with the integrals over one period, the inequality preceding (3.6) adapts to periodic functions, and consequently the *a priori* estimates (3.6) and (3.7) also hold for periodic solutions of (3.2).

Consider the Sobolev space $H^1(-l,l)$ of (equivalence classes of) even real functions, with norm given by

$$\|u\|_1^2 = \int_l^l (u^2+\beta u_x^2)\,\mathrm{d}x. \tag{4.1}$$

The functional $F = \frac{1}{2}\|\cdot\|_0^2$, say, is weakly continuous in this space (according to the famous theorem named after Rellich). Because there is a positive constant μ such that $\inf u(x) \geqslant -\mu\|u\|_1$, the functional G is bounded below in $H^1(-l,l)$; specifically,

$$\begin{aligned} G(u) &\geqslant -(\tfrac{1}{3}\mu\|u\|_1+\tfrac{1}{2}|C_{\mathrm{m}}|)\|u\|_0^2 \\ &\geqslant -(\tfrac{1}{3}\mu\|u\|_1+\tfrac{1}{2}|C_{\mathrm{m}}|)\|u\|_1^2. \end{aligned}$$

Moreover, being expressible as the sum $\frac{1}{2}\|u\|_1^2$ plus weakly continuous functionals, G is *weakly lower semicontinuous* in the space $H^1(-l,l)$.

It follows at once that $\min G(u)$ for given $F(u) > 0$ is achieved by an element $\phi(x;2l) \in H^1(-l,l)$.

There remains only to confirm that the respective constant solution $\phi = A = -(2F/l)^{\frac{1}{2}}$ of (3.2), with $C = A$, is not the conditional minimizer. In the expression (3.12) for the second variation, take $\xi = \cos k_{\mathrm{m}} x$, which, because $l = N\pi/k_{\mathrm{m}}$, is an admissible even periodic variation satisfying (3.13). The result is

$$S(A\,;\cos k_{\mathrm{m}} x) = \tfrac{1}{2}\epsilon^2 l(A+C_{\mathrm{m}}) < 0,$$

showing that $G(A)$ is not the conditional minimum. (In fact, the trivial solution A may be seen not to minimize G for given F whenever $l > 2\pi\beta/\{\alpha+(\alpha^2-4\beta A)^{\frac{1}{2}}\}$.)

Note incidentally how for periodic solutions the estimate corresponding to (3.7) also follows from the minimizing property of $\phi(x;2l)$. With $a = \pm(2F/l)^{\frac{1}{2}}$, we have $u = a\cos k_{\mathrm{m}} x$ as a competitor for the conditional minimum of G; but it is an unsuccessful competitor in not satisfying (3.2). Hence

$$G(\phi) < G(a\cos k_{\mathrm{m}} x) = C_{\mathrm{m}} F < 0.$$

For the periodic case, the identity (3.5) is equivalent to

$$CF(\phi) = G(\phi)+\frac{1}{6}\int_{-l}^{l}\phi^3\,\mathrm{d}x,$$

where the explicit integral must be non-positive and is presumably negative (because, if it were positive, $G(-\phi)$ would be less than $G(\phi)$, contrary to the minimizing property of ϕ). Hence $C < C_{\mathrm{m}}$.

4.2. *Regularity*

Having been established as the minimizer of G for given $F > 0$, the equivalence class $\phi(x; 2l) \in H^1(-l, l)$ satisfying (3.2) weakly is now shown in fact to include an infinitely differentiable function. Writing $\mathscr{A} = (|C| - \beta\partial_x^2)^{-1}$, let us recast (3.2) in the form

$$\phi = -\mathscr{A}(\phi^2 - \alpha \mathrm{L}\phi), \tag{4.2}$$

which is an operator equation acting in $H^1(-l, l)$. With reference to the Fourier-series representation of any $u \in H^1(-l, l)$, i.e. to

$$u(x) = \sum_{n=-\infty}^{\infty} a_n \cos k_n x \tag{4.3}$$

with $k_n = n\pi/l$ and $a_{-n} = a_n$, we have

$$\mathscr{A}u(x) = \sum_{n=-\infty}^{\infty} \frac{1}{|C| + \beta k_n^2} a_n \cos k_n x. \tag{4.4}$$

That is, the symbol of $\mathscr{A}$ is $(|C| + \beta k_n^2)^{-1}$. Similarly, the symbol of $\mathscr{A}\mathrm{L}$ is $|k_n|/(|C| + \beta k_n^2)$.

As a generalization of (4.1), higher-order Sobolev spaces $\mathscr{H}^s(-l, l)$ with $s = 2, 3, \ldots,$ have norms given by

$$\begin{aligned} \|u\|_s^2 &= 2l \sum_{n=-\infty}^{\infty} (1 + \beta k_n^2)^s a_n^2 \\ &= \int_{-l}^{l} \left\{ \sum_{j=0}^{s} \binom{s}{j} \beta^j (\partial_x^j u)^2 \right\} \mathrm{d}x, \end{aligned} \tag{4.5}$$

with $$\binom{s}{j} = \frac{s!}{j!(s-j)!}.$$

According to the Riemann-Lebesgue theorem referred to (4.3) and (4.5), the attribution $u \in H^s(-l, l)$ implies $\partial_x^{s-1} u$ to be a continuous function (or rather its equivalence class includes such a function). In particular, $u \in H^1(-l, l)$ implies $u(x)$ to be continuous, a fact already evoked in the second sentence after (4.1).

The original attribution $\phi \in H^1(-l, l)$ therefore implies the same for ϕ^2. Hence (4.4) and (4.5) show that $\mathscr{A}\phi^2 \in H^3(l, l)$. Also $\mathscr{A}\mathrm{L}\phi \in H^2(-l, l)$. It thus follows from (4.2) that $\phi \in H^2(-l, l)$, and by induction the same argument leads to the conclusion that $\phi \in H^\infty(-l, l)$. Thus the conditional minimum of G is proven to be realized by a C^∞ function $\phi(x; 2l)$ satisfying (3.2).

4.3. *Passage to solitary waves*

For each $l = N\pi/k_{\mathrm{m}}$ with $N = 1, 2, 3, \ldots,$ a non-trivial smooth periodic solution of (3.2) has been shown to exist corresponding to a given positive value of F, defined by an integral (3.8) from $-l$ to l. Each of these periodic solutions satisfies the *a priori* estimates (3.4), (3.6), (3.7), (3.15) and (3.16) when the integrals are redefined in the same way. Evidently, to maintain the same F when $N \to \infty$, $\phi^2(x; 2l)$ must tend to zero in subsets of increasing large measure within the interval $[-l, l]$, for instance in

$(\frac{1}{2}l, l)$ and $(-l, -\frac{1}{2}l)$. It can therefore be concluded that, corresponding to each given $F > 0$, (3.2) has a smooth solitary-wave solution $\phi(x)$ such that $|\phi(x) - \phi(x; 2l)| \to 0$ in $(-\frac{1}{2}l, \frac{1}{2}l)$ as $l = N\pi/k_m \to \infty$.

4.4. *Other approaches*

In the past various other methods have been used to establish the existence of solitary-wave solutions of nonlinear problems where such solutions are not known explicitly. For example, a general method based on positive-operator theory was developed by Benjamin, Bona & Bose (1990) for the treatment of problems where solitary waves belong to a cone of non-negative functions defined on $\boldsymbol{R}$. But this method seems to be unavailing for the present problem, solitary-wave solutions of which are always oscillatory functions.

The following alternative is noteworthy, however. In terms of the Fourier transform $\hat{\phi}(k) = \mathscr{F}\phi$, which is an even real function if ϕ is even, equation (3.2) is equivalent to

$$\hat{\phi}(k) = -\frac{(\hat{\phi} * \hat{\phi})(k)}{\beta k^2 - \alpha|k| - C}, \tag{4.6}$$

where $\hat{\phi} * \hat{\phi}$ denotes the convolution of $\hat{\phi}$ with itself. For periodic solutions of (3.2), an equation akin to (4.6) holds for the set of real coefficients $\{a_n\}_{n=-\infty}^{\infty}$ in the Fourier-series representation (4.3) of $\phi(x; 2l)$.

When C is prescribed to satisfy $C < C_m$, the denominator on the right-hand side of (4.6) is positive for all $k \in \boldsymbol{R}$. It thus appears likely that the solitary-wave solution $\phi(x)$ of (3.2) for any given $C < C_m$ has a non-positive Fourier transform (i.e. $\hat{\phi}(k) \leqslant 0 \, \forall \, k \in \boldsymbol{R}$). Again, periodic solutions $\phi(x; 2l)$ with l large enough may have Fourier (cosine) series with non-positive coefficients. An existence theory could be directed to (4.6) as an operator equation posed in a cone of non-positive functions, or to the corresponding equation for Fourier coefficients posed in a cone of non-positive sequences (i.e. sequences $\{b_n\}_{n=-\infty}^{\infty} \in l_2$, say, all of whose elements satisfy $b_n \leqslant 0$).

5. Conclusion

About the new class of oscillatory solitary waves that has been investigated here, the following point deserves re-emphasis. Because they minimize the functional G for given values of the functional F, and because G and F are invariants for any solutions of the evolution equation (1.1) or (2.8), these waves are likely to be *orbitally stable* (i.e. Lyapunov stable with respect to a metric that factors out translations). The property that solitary-wave solutions of the KdV equation are conditional minimizers (or maximizers, according to the sign of β) of invariant functionals has been used by Benjamin (1972) and Bona (1975) to prove the orbital stability of such solutions. And many others since have adapted the method to solitary-wave solutions of comparable evolution equations. Although now deferred, a proof of stability should be feasible in the present case too.

It is therefore reasonable to expect that solitary waves of the present type should be observable experimentally, although h will probably need to be quite small. The condition (2.3) is the main criterion of realizability. Take, for example, the case of benzene on water. Then $\rho_2 - \rho_1 = 0.3$ g cm^{-3} and $T = 35$ dyn cm^{-1} (Kaye & Laby 1966, p. 42). It follows that $2T/g(\rho_2 - \rho_1)h^2 = 24/h^2$ with h in mm. For $h = 2$ mm, say, the value 6 of the dimensionless number in question is perhaps large enough for confidence in the relevance of the present theory. Even better experimental prospects may be provided by other organic liquids, specifically ones that are immiscible with water and have significant surface tensions against water, also having fairly small viscosities and densities only slightly less than that of water.

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